SPLITTING, PARALLEL GRADIENT AND BAKRY-EMERY RICCI CURVATURE

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To my beloved granddaughters, Júlia and Clara

Abstract. In this paper we obtain a splitting theorem for the symmetric diffusion operator $\Delta_\phi = \Delta - \langle\nabla\phi, \nabla\rangle$ and a non-constant C^3 function f in a complete Riemannian manifold M, under the assumptions that the Ricci curvature associated with Δ_ϕ satisfies $\mathrm{Ric}_\phi(\nabla \mathrm{f}, \nabla \mathrm{f}) \geq 0$, that $|\nabla f|$ attains a maximum at M and that Δ_ϕ is non-decreasing along the orbits of ∇f. The proof uses the general fact that a complete manifold M with a non-constant smooth function f with parallel gradient vector field must be a Riemannian product $M = N \times \mathbb{R}$, where N is any level set of f.

1. **Introduction**

Several papers obtained splitting theorems on complete Riemannian manifolds (M, g) assuming non-negative sectional curvature, non-negative Ricci curvature or non-negative Bakry-Emery Ricci curvature, in the presence of some line in M (see for example [T], [CG], [EH], [FLZ], [WW]). In all these papers the Busemann function b_γ associated with a ray γ is studied. In general it is proved that the assumptions imply that b_γ is smooth and has parallel gradient vector field. In this paper we will not assume the existence of a line.

We will consider the symmetric diffusion operator $\Delta_\phi u = \Delta u - \langle\nabla\phi, \nabla u\rangle$, where Δ is the Laplace-Beltrami operator and ϕ is a given C^2 function on M. The operator Δ_ϕ is used in probability theory, potential theory and harmonic analysis on complete and non-compact Riemannian manifolds. Another important motivation is that, when Δ_ϕ is seen as a symmetric operator in $L^2(M, e^{-\phi}dv_g)$, it is unitarily equivalent to the Schrödinger operator $\Delta - \frac{1}{4}|\nabla\phi|^2 + \frac{1}{2}\Delta_\phi$ in $L^2(M, dv_g)$, where dv_g is the volume element of (M, g) (see for example [D], [W], [L]).

Let n be the dimension of M. For $m \in [n, +\infty]$ we follow [L] and define the m-dimensional Ricci curvature Ric_{mn} associated with the operator Δ_ϕ as follows. Set $\mathrm{Ric}_{nn} = \mathrm{Ric}$, where Ric is the usual Ricci curvature. If $n < m < \infty$ set

$$\mathrm{Ric}_{mn}(X, X) = \mathrm{Ric}(X, X) + \mathrm{Hess}(\phi)(X, X) - \frac{|\langle\nabla\phi, X\rangle|^2}{m - n}.$$

Finally set $\mathrm{Ric}_{\infty n} = \mathrm{Ric}_\phi = \mathrm{Ric} + \mathrm{Hess}(\phi)$.

Now we can state our first result:

Theorem A. *Let M be a complete connected Riemannian manifold. Assume that there exist a C^3 function f and a C^2 function ϕ on M satisfying the following conditions:*

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(1) $\mathrm{Ric}_\phi(\nabla \mathrm{f}, \nabla \mathrm{f}) \geq 0$*;*
(2) $|\nabla f|$ *has a positive global maximum;*
(3) $\Delta_\phi f$ *is non-decreasing along the orbits of* ∇f*.*

Then f *is smooth and* M *is isometric to the Riemannian product* $N \times \mathbb{R}$*, where* N *is any level set of* f*. Furthermore it holds that* ϕ *and* f *are affine functions on each fiber* $\{x\} \times \mathbb{R}$*.*

Remark 1. We will see in Section 4 that each one of conditions (1), (2), (3) is essential in Theorem A.

By a similar proof it can be proved a local version for Theorem A:

Theorem B. *Let* M *be a connected Riemannian manifold. Assume that there exist a* C^3 *function* f *and a* C^2 *function* ϕ *on* M *satisfying the following conditions:*
(1) $\mathrm{Ric}_\phi(\nabla \mathrm{f}, \nabla \mathrm{f}) \geq 0$*;*
(2) $|\nabla f|$ *has a positive global maximum on* M*;*
(3) $\Delta_\phi f$ *is non-decreasing along the orbits of* ∇f*.*

Then f *is smooth and for each point* $p \in M$ *there exist* $\epsilon > 0$ *and an open neighborhood* V *of* p *such that* $V = N \times (-\epsilon, \epsilon)$*, where* N *is some level set of* $f|_V$*. Furthermore it holds that* ϕ *and* f *are affine functions on each fiber* $\{x\} \times (-\epsilon, \epsilon)$*.*

By applying Theorem B to some neighborhood of a point p where $|\nabla f|$ has a local maximum, we obtain:

Corollary 1.1. *Let* M *be a connected Riemannian manifold. Assume that there exist a* C^3 *function* f *and a* C^2 *function* ϕ *on* M *satisfying the following conditions:*
(1) $\mathrm{Ric}_\phi(\nabla \mathrm{f}, \nabla \mathrm{f}) \geq 0$*;*
(2) $|\nabla f|$ *has a positive local maximum at some point* $p \in M$*;*
(3) $\Delta_\phi f$ *is non-decreasing along the orbits of* ∇f *in a neighborhood of* p*.*

Then there exist $\epsilon > 0$ *and an open neighborhood* V *of* p *such that* $f|_V$ *is smooth and* $V = N \times (-\epsilon, \epsilon)$*, where* N *is some level set of* $f|_V$*. Furthermore it holds that* ϕ *and* f *are affine functions on each fiber* $\{x\} \times (-\epsilon, \epsilon)$*.*

Remark 2. Since $\mathrm{Ric}_\phi(\nabla \mathrm{f}, \nabla \mathrm{f}) \geq \mathrm{Ric}_{\mathrm{mn}}(\nabla \mathrm{f}, \nabla \mathrm{f})$, for all $m \in [n, +\infty]$, Theorems A, B and Corollary 1.1 also hold if we replace the condition $\mathrm{Ric}_\phi(\nabla \mathrm{f}, \nabla \mathrm{f}) \geq 0$ by the assumption $\mathrm{Ric}_{\mathrm{mn}}(\nabla \mathrm{f}, \nabla \mathrm{f}) \geq 0$.

The main fact that will be used in the proof of Theorem A is the following simple general result, which does not require curvature conditions or the existence of lines.

Proposition 1.1. *Let* M *be a complete connected Riemannian manifold. Assume that there exists a non-constant smooth function* $f : M \to \mathbb{R}$ *such that* ∇f *is a parallel vector field. Then* M *is isometric to* $N \times \mathbb{R}$*, where* N *is any level set of* f*. Furthermore* f *must be an affine function on each fiber* $\{x\} \times \mathbb{R}$*. More precisely, if* $N = f^{-1}(\{c\})$ *and* $|\nabla f| = C$*, the obtained isometry* $\varphi : N \times \mathbb{R} \to M$ *maps each fiber* $\{x\} \times \mathbb{R}$ *onto the image of the orbit of* ∇f *which contains* x*, and it holds that* $(f \circ \varphi)(x, t) = c + Ct$*.*

2. **Functions with parallel gradient vector field**

We recall that a smooth vector field X in M is said to be parallel if for any point $p \in M$, any open neighborhood U of p, and any smooth vector field Y in U, it holds that $\big(\nabla_Y X\big)(p) = 0$.

Proposition 1.1 could be proved by using the de Rham Decomposition Theorem on the universal cover of M with the induced metric. However, we preferred to present a more elementary proof which just uses the following Berger's extension of Rauch's Comparison Lemma (see for example [CE]).

Lemma 2.1 (Berger)**.** *Consider complete Riemannian manifolds $W, \tilde{W}$ whose dimensions satisfy $\dim(W) \geq \dim(\tilde{W})$, a smooth positive function $g : [a,b] \to \mathbb{R}$, unit speed geodesics $\gamma : [a,b] \to W$, $\tilde{\gamma} : [a,b] \to \tilde{W}$, and unit parallel vector fields E along γ and $\tilde{E}$ along $\tilde{\gamma}$, satisfying $\langle E, \gamma' \rangle = \langle \tilde{E}, \tilde{\gamma}' \rangle = 0$. For $\delta > 0$ and $(s,u) \in [a,b] \times [0,\delta]$, set $\psi_s(u) = \psi^u(s) = \exp_{\gamma(s)} ug(s)E(s)$ and $\tilde{\psi}_s(u) = \tilde{\psi}^u(s) = \exp_{\tilde{\gamma}(s)} ug(s)\tilde{E}(s)$. Assume that, for any $s \in [a,b]$, the geodesic $\psi_s : [0,\delta] \to W$ is free of focal points with respect to $\psi_s(0) = \gamma(s)$. Assume further that, for any $(s,u) \in [a,b] \times [0,\delta]$, any unit vector $v \in T_{\psi_s(u)}W$ with $\langle v, \psi_s'(u) \rangle = 0$, and any unit vector $\tilde{v} \in T_{\tilde{\psi}_s(u)}\tilde{W}$ with $\langle \tilde{v}, \tilde{\psi}_s'(u) \rangle = 0$, the sectional curvatures satisfy*

$$K\big(v, \psi_s'(u)\big) = \big\langle R\left(v, \psi_s'(u)\right) \psi_s'(u), v \big\rangle \geq \big\langle \tilde{R}\big(\tilde{v}, \tilde{\psi}_s'(u)\big) \tilde{\psi}_s'(u), \tilde{v} \big\rangle = \tilde{K}\big(\tilde{v}, \tilde{\psi}_s'(u)\big),$$

where $R, \tilde{R}$ are the corresponding tensor curvatures of W, respectively, $\tilde{W}$. Then it holds that the length $L(\psi^u) \leq L\big(\tilde{\psi}^u\big)$, for any $u \in [0,\delta]$.

Remark 3. In the statement of the above Berger's Lemma in [CE], it was assumed that $K(\mu,\nu) \geq \tilde{K}(\tilde{\mu},\tilde{\nu})$ for any orthonormal vectors $\mu, \nu \in T_xW$, any orthonormal vectors $\tilde{\mu}, \tilde{\nu} \in T_{\tilde{x}}\tilde{W}$ and any $x \in W$, $\tilde{x} \in \tilde{W}$. However, the same proof as in [CE] may be used to prove the more general formulation as in Lemma 2.1 above.

Consider a C^1 function g on a manifold such that $|\nabla g|$ is a constant D. Let μ be an orbit of ∇g. We recall the following simple well-known equality:

$$\begin{aligned} g\big(\mu(t)\big) &= g\big(\mu(a)\big) + \int_a^t (g \circ \mu)'(s) ds = g\big(\mu(a)\big) + \int_a^t \big\langle \nabla g\big(\mu(s)\big), \mu'(s) \big\rangle ds \\ &= g\big(\mu(a)\big) + \int_a^t \big|\nabla g\big(\mu(s)\big)\big|^2 ds = g\big(\mu(a)\big) + D^2 (t-a). \end{aligned} \tag{1}$$

Proof of Proposition 1.1. Let M be a complete Riemannian manifold and assume that there exists a non-constant smooth function f such that ∇f is parallel. In particular $|\nabla f|$ is a constant $C > 0$.

Claim 2.1. *Fix $p \in M$ and a unit vector field X in a neighborhood of p which is orthogonal to $\nabla f(p)$ at p. Then the sectional curvature*

$$K\left(X(p), \frac{\nabla f(p)}{C}\right) = 0.$$

In fact, since ∇f is parallel we have that

$$\big(\nabla_X \nabla_{\nabla f} \nabla f - \nabla_{\nabla f} \nabla_X \nabla f - \nabla_{[X,\nabla f]} \nabla f\big)(p) = 0,$$

which proves Claim 2.1.

From now on we fix a level set $N = f^{-1}(\{c\}) \subset M$, for some $c \in \mathbb{R}$.

Claim 2.2. *N is a totally geodesic embedded hypersurface.*

Indeed, since ∇f has no singularities, the local form of submersions imply that N is a smooth embedded hypersurface. Fix $p \in N$ and a geodesic σ in M satisfying $\sigma(0) = p$ and $\langle \nabla f(\sigma(0)), \sigma'(0) \rangle = 0$. Since ∇f and σ' are parallel vector fields along

σ, we obtain that $(f\circ\sigma)'(s) = \langle \nabla f(\sigma(s)), \sigma'(s)\rangle = \langle \nabla f(\sigma(0)), \sigma'(0)\rangle = 0$ for all s, hence the image of σ is contained in the level set N, which shows that N is totally geodesic. Claim 2.2 is proved.

Note that the orbits of ∇f intersect N orthogonally and do not intersect each other. Furthermore they are geodesics, since ∇f is parallel along them. In particular the normal exponential map $\exp^{\perp} : TN^{\perp} \to M$ is injective. It is also surjective, since, for each point $p \in M$, Equation (1) above implies that the orbit ν of ∇f which contains p satisfies $(f\circ\gamma)(\mathbb{R}) = \mathbb{R}$, hence ν must intersect (orthogonally) the level set $f^{-1}(\{c\}) = N$. We conclude that $\exp^{\perp}$ is a diffeomorphism. Thus we define the map $\varphi : N\times\mathbb{R} \to M$ given by

$$\varphi(x,t) = \exp_x \frac{t\,\nabla f(x)}{C} = \exp^{\perp}\left(x, \frac{t\,\nabla f(x)}{C}\right) = \exp^{\perp}\left(x, \frac{t\,\nabla f(x)}{|\nabla f(x)|}\right).$$

Since $\exp^{\perp}$ is a diffeomorphism, it is easy to see that φ is also a diffeomorphism.

Let P_t denote the parallel transport along the unit speed geodesic $\mu(t) = \varphi(x,t)$. By using the fact that ∇f is parallel along this geodesic, we obtain that

$$\frac{\partial\varphi}{\partial t}(x,t) = \mu'(t) = P_t\big(\mu'(0)\big) = P_t\left(\frac{\partial\varphi}{\partial t}(x,0)\right) = P_t\left(\frac{\nabla f(x)}{C}\right) = \frac{\nabla f(\varphi(x,t))}{C}. \tag{2}$$

In particular μ is an orbit of the unit vector field $\nabla\left(\frac{f}{C}\right)$. Applying (1) to the function $g = \frac{f}{C}$, we obtain that $\left(\frac{f}{C}\right)(\varphi(x,t)) = \left(\frac{f}{C}\right)(\mu(t)) = \left(\frac{f}{C}\right)(\mu(0)) + t = \frac{c}{C} + t$, hence

$$f\big(\varphi(x,t)\big) = c + Ct, \text{ for any } x\in N \text{ and any } t\in\mathbb{R}. \tag{3}$$

Since φ is a diffeomorphism, to prove that M is isometric to $N\times\mathbb{R}$, we just need to prove that $d\varphi_{(x,t)} : T_{(x,t)}(N\times\mathbb{R}) \to T_{\varphi(x,t)}M$ is a linear isometry for any $(x,t)\in N\times\mathbb{R}$. To do this, we will fix $(x,t)\in N\times\mathbb{R}$ and will consider first the curve $\alpha(s) = (x, t+s)$, which satisfies $\alpha(0) = (x,t)$ and $|\alpha'(0)| = 1$. Then we will show that $|(\varphi\circ\alpha)'(0)| = 1 = |\alpha'(0)|$. We will also consider any unit speed geodesic β orthogonal to α at (x,t). We will show that $\langle(\varphi\circ\alpha)'(0), (\varphi\circ\beta)'(0)\rangle = 0 = \langle\alpha'(0), \beta'(0)\rangle$ and $|(\varphi\circ\beta)'(0)| = 1 = |\beta'(0)|$. Then we will conclude that $d\varphi_{(x,t)} : T_{(x,t)}(N\times\mathbb{R}) \to T_{\varphi(x,t)}M$ is a linear isometry and φ is an isometry.

By derivating the equality $(\varphi\circ\alpha)(s) = \varphi(x, t+s)$ and using (2) we obtain

$$(\varphi\circ\alpha)'(s) = \frac{\partial\varphi}{\partial s}(x, t+s) = \frac{\nabla f\big(\varphi(x,t+s)\big)}{C} = \frac{\nabla f\big((\varphi\circ\alpha)(s)\big)}{C}. \tag{4}$$

In particular it holds that

$$|(\varphi\circ\alpha)'(0)| = \left|\frac{\nabla f\big(\varphi(x,t)\big)}{C}\right| = 1 = |\alpha'(0)|. \tag{5}$$

Fix $\epsilon > 0$. Consider a unit speed geodesic $\beta : [-\epsilon,\epsilon] \to M$ satisfying $\beta(0) = (x,t) = \alpha(0)$, $|\beta'(0)| = 1$ and $\langle\alpha'(0), \beta'(0)\rangle = 0$. Since $\beta'(0)$ is tangent to the totally geodesic submanifold $N\times\{t\}$, we may write $\beta(s) = (\eta(s), t)$ where $\eta : [-\epsilon,\epsilon] \to N$ is a geodesic in N satisfying $\eta(0) = x$ and $|\eta'(0)| = 1$. Note that η is also a geodesic in M, since N is totally geodesic by Claim 2.2. By using (3) we obtain that

$$f\big((\varphi\circ\beta)(s)\big) = f\big(\varphi(\eta(s),t)\big) = c + Ct. \tag{6}$$

As a consequence it holds that

$$(\varphi\circ\beta)\big([-\epsilon,\epsilon]\big)\subset f^{-1}\big(\{c+Ct\}\big). \tag{7}$$

From (4), (7) and the equality $\alpha(0)=\beta(0)=(x,t)$, we obtain that

$$\begin{aligned}\langle(\varphi\circ\beta)'(0),(\varphi\circ\alpha)'(0)\rangle &= \left\langle(\varphi\circ\beta)'(0),\frac{\nabla f\big((\varphi\circ\alpha)(0)\big)}{C}\right\rangle\\ &= \left\langle(\varphi\circ\beta)'(0),\frac{\nabla f\big((\varphi\circ\beta)(0)\big)}{C}\right\rangle = 0 = \langle\beta'(0),\alpha'(0)\rangle\,. \end{aligned}\tag{8}$$

Now we consider the unit speed geodesic $\beta_0(s)=\big(\eta(s),0\big)$. Let E be the unit parallel vector field along β_0 which is orthogonal to $N\times\{0\}$ and satisfies $\big(\eta(s),u\big)=\exp_{\beta_0(s)}uE(s)$, for any $u\in\mathbb{R}$. Set $\psi_s(u)=\psi^u(s)=\exp_{\beta_0(s)}uE(s)$. In particular we have that

$$\psi_s(t)=\psi^t(s)=\exp_{\beta_0(s)}tE(s)=(\eta(s),t)=\beta(s). \tag{9}$$

Since ∇f is parallel and N is totally geodesic, the vector field $\tilde{E}(s)=\frac{\nabla f(\eta(s))}{C}$ is a unit parallel vector field along η which is orthogonal to N. Set $\tilde{\psi}_s(u)=\tilde{\psi}^u(s)=\exp_{\eta(s)}u\tilde{E}(s)=\varphi\big(\eta(s),u\big)$. Thus we obtain from (2) that

$$\tilde{\psi}_s'(u)=\frac{\partial\varphi}{\partial u}\big(\eta(s),u\big)=\frac{\nabla f\big(\varphi(\eta(s),u)\big)}{C}=\frac{\nabla f\big(\tilde{\psi}_s(u)\big)}{C}. \tag{10}$$

Note also that

$$\tilde{\psi}_s(t)=\tilde{\psi}^t(s)=\varphi\big(\eta(s),t\big)=(\varphi\circ\beta)(s). \tag{11}$$

To compare curvatures, we fix $s\in[-\epsilon,\epsilon]$, $u\geq 0$, and unit vectors $v\in T_{\psi_s(u)}(N\times\mathbb{R})$ and $\tilde{v}\in T_{\tilde{\psi}_s(u)}M$ satisfying $\langle v,\psi_s'(u)\rangle=0=\big\langle\tilde{v},\tilde{\psi}_s'(u)\big\rangle$. By using the Riemannian product $N\times\mathbb{R}$, Claim 2.1 and Equation (10), we obtain that

$$K\big(v,\psi_s'(u)\big)=0=\ K\left(\tilde{v},\frac{\nabla f\big(\tilde{\psi}_s(u)\big)}{C}\right)=K\big(\tilde{v},\tilde{\psi}_s'(u)\big). \tag{12}$$

Since $\tilde{\psi}_s$ is a geodesic orthogonal to N and $\exp^\perp:TN^\perp\to M$ is a diffeomorphism, we have that $\tilde{\psi}_s$ is free of focal points to $\tilde{\psi}_s(0)$. Similarly we have that ψ_s is free of focal points to $\psi_s(0)$. From (9) and (11) we have that $\psi^t=\beta$ and $\tilde{\psi}^t=\varphi\circ\beta$. By using (12) we may apply Lemma 2.1 with $W=N\times\mathbb{R}$, $\tilde{W}=M$, $g=1$, $\gamma=\beta_0$, $\tilde{\gamma}=\eta$, obtaining that $L(\beta)=L(\psi^t)\leq L(\tilde{\psi}^t)=L(\varphi\circ\beta)$. We apply this lemma again with $W=M$, $\tilde{W}=N\times\mathbb{R}$, $g=1$, $\gamma=\eta$, $\tilde{\gamma}=\beta_0$, obtaining that $L(\varphi\circ\beta)\leq L(\beta)$. Varying $\epsilon>0$ we conclude that

$$|(\varphi\circ\beta)'(0)|=|\beta'(0)|\,. \tag{13}$$

From (5), (8), (13) we obtain that $d\varphi_{(x,t)}:T_{(x,t)}(N\times\mathbb{R})\to T_{\varphi(x,t)}M$ is a linear isometry. Thus the diffeomorphism $\varphi:N\times\mathbb{R}\to M$ is an isometry. Furthermore we have from (3) that $(f\circ\varphi)(x,t)=c+Ct$, hence $f\circ\varphi$ is an affine function on the fiber $\{x\}\times\mathbb{R}$. Proposition 1.1 is proved. $\square$

A similar proof as above proves the following local version for Proposition 1.1:

Proposition 2.1. *Let M be a connected Riemannian manifold. Assume that there exists a non-constant smooth function f on an open subset U, such that ∇f is a parallel vector field on U. Then, for each point $p \in U$, there exist $\epsilon > 0$ and an open neighborhood $V \subset U$ of p such that V is isometric to $N \times (-\epsilon, \epsilon)$, where N is a smooth level set of $f|_V$. Furthermore f must be an affine function on each fiber $\{x\} \times (-\epsilon, \epsilon)$.*

3. Proof of Theorems A and B

To prove Theorems A and B we first recall the famous Bochner formula:

$$\frac{1}{2}\Delta|\nabla f|^2 = \mathrm{Ric}(\nabla \mathrm{f}, \nabla \mathrm{f}) + \nabla \mathrm{f}(\Delta \mathrm{f}) + \sum_{\mathrm{i}=1}^{\mathrm{n}} |\nabla_{\mathrm{E_i}} \nabla \mathrm{f}|^2, \tag{14}$$

where n is the dimension of the Riemannian manifold M. By using (14), a direct calculation leads easily to the generalized Bochner formula below (see [L]):

$$\frac{1}{2}\Delta_\phi|\nabla f|^2 = \mathrm{Ric}_\phi(\nabla \mathrm{f}, \nabla \mathrm{f}) + \nabla \mathrm{f}(\Delta_\phi \mathrm{f}) + \sum_{\mathrm{i}=1}^{\mathrm{n}} |\nabla_{\mathrm{E_i}} \nabla \mathrm{f}|^2. \tag{15}$$

Assume that the hypotheses of Theorem A hold. Fix $p \in M$ and a local unit vector field X in an open normal ball B centered at p. Set $X = E_1$ and construct a local orthonormal frame $E_1, \cdots, E_n$ in B. By the hypotheses of Theorem A, each parcel on the right side of (15) is nonnegative, hence $\Delta_\phi|\nabla f|^2 \geq 0$. Since $|\nabla f|$ assumes a global maximum we conclude that $|\nabla f|$ is constant by the maximum principle for elliptic linear operators (see Lemma 2.4 in [FLZ]). The fact that $\Delta_\phi|\nabla f|^2 = 0$ implies that each parcel on the right side of (15) vanishes. In particular $\sum_{i=1}^n |\nabla_{E_i}\nabla f|^2 = 0$, hence $\nabla_X \nabla f = \nabla_{E_1}\nabla f = 0$. Since p and X were chosen arbitrarily, we obtain that ∇f is parallel. In particular f is smooth. By Proposition 1.1, there exists an isometry $\varphi : N \times \mathbb{R} \to M$, where N is some fixed level set of f, and f is an affine function along each fiber $\{x\} \times \mathbb{R}$.

By Claim 2.1 in the proof of Proposition 1.1, we have that $\mathrm{Ric}(\nabla f, \nabla f) = 0$. Since $\mathrm{Ric}_{\,\phi}(\nabla f, \nabla f) = 0$ we obtain that $\mathrm{Hess}(\phi)(\nabla f, \nabla f) = 0$. Thus the fact that $\nabla_{\nabla f}\nabla f = 0$ implies easily that $\nabla f\big(\nabla f(\phi)\big) = 0$, hence ϕ is an affine function along any orbit $\varphi\big(\{x\} \times \mathbb{R}\big)$ of ∇f. The proof of Theorem A is complete.

Theorem B is proved by using Proposition 2.1 and proceeding similarly as in the proof of Theorem A.

4. Examples

In this section we will see that each one of conditions (a), (b), (c) in Theorem A is essential, even in the case that ϕ is constant.

Example 1. *Let M be the hyperbolic n-dimensional space and f the Busemann function associated to some ray γ in M. We know that any orbit σ of ∇f is a line containing a ray asymptotic to γ. It is also known that $|\nabla f| = 1$ and that $\Delta f = n - 1$ (see [CM]), hence conditions (b) and (c) in Theorem A hold. Thus condition (a) is essential in Theorem A.*

Example 2. *Consider a smooth curve $\alpha : \mathbb{R} \to \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ such that $\alpha(t) = \big(t, g(t), 0\big)$ if $t \in (-1, 1)$, where g is an even strict convex nonnegative smooth function satisfying $g(0) = 0 = g'(0)$ and $\lim_{|t|\to 1} g(t) = 1$. Assume further that $\alpha(t) = (1, t, 0)$ if $t \geq 1$ and $\alpha(t) = (-1, |t|, 0)$ if $t \leq -1$. Let M be the smooth*

surface in $\mathbb{R}^3$ *obtained rotating the image of* α *around the* y *axis. Clearly the Gauss curvature of* M *is nonnegative. Consider the function* $F(x,y,z) = y$ *and let* f *be the restriction of* F *to* M*. Note that* $x^2 + z^2 \leq 1$*, and* $|\nabla f(p)| = 1$ *if* $p = (x,y,z) \in M$ *and* $y \geq 1$*. Since* $|\nabla f|$ *is constant outside a compact set, we have that* $|\nabla f|$ *attains a maximum at some point of* M*. Thus* M *satisfies conditions (a) and (b) in Theorem A. This shows that condition (c) is essential in this theorem.*

Example 3. *Consider the paraboloid* $M \subset \mathbb{R}^3$ *given by the equation* $z = x^2 + y^2$ *and the function* $F : \mathbb{R}^3 \to \mathbb{R}$ *given by* $F(x,y,z) = z^2$*. Set* $f = F|_M$*. Since* M *has positive Gauss curvature, condition (a) in Theorem A holds. In* $M - \{(0,0,0)\}$ *we consider the coordinates* $\varphi(\rho,\theta) = \big(\rho\cos\theta, \rho\sin\theta, \rho^2\big)$ *for* $\rho > 0$*. On these coordinates we have* $\varphi_\rho = (\cos\theta, \sin\theta, 2\rho)$ *and* $\nabla f = \frac{4\rho^3}{1+4\rho^2}\,\varphi_\rho$*, hence* $\big|\nabla f\big(\varphi(\rho,\theta)\big)\big| = \frac{4\rho^3}{\sqrt{1+4\rho^2}}$*. As a consequence* ∇f *is unbounded, hence condition (b) in Theorem A fails. Now we will see that condition (c) holds. Since the orbit of* ∇f *at* $(0,0,0)$ *is trivial, we just need to check that condition (c) holds in* $M - \{(0,0,0)\}$*. A direct computation leads us to*

$$\Delta f = \frac{4\rho^2(3 + 8\rho^2)}{(1+4\rho^2)^2} + \frac{4\rho^2}{1+4\rho^2}\,.$$

Thus we obtain that

$$\varphi_\rho(\Delta f) = \frac{d}{d\rho}\,(\Delta f) = \frac{8\rho(3+4\rho^2)}{(1+4\rho^2)^3} + \frac{8\rho}{(1+4\rho^2)^2} > 0. \tag{16}$$

Since $\nabla f = \frac{4\rho^3}{1+4\rho^2}\,\varphi_\rho$ *we have from (16) that* $\nabla f(\Delta f) > 0$ *in* $M - \{(0,0,0)\}$*, hence* Δf *is non-decreasing along the orbits of* ∇f*. Thus condition (c) holds, which shows that condition (b) is essential in Theorem A.*

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